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COMPONENT GROUP OF THE p-NEW SUBVARIETY OF $J_0(Mp)$

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ABSTRACT

For an abelian variety A over $\mathbb{Q}_p$, the special fibre in the Néron model of A over $\mathbb{Z}_p$ is the extension of a finite group scheme over $\mathbb{F}_p$, called the group of connected components, by the connected component of identity. When A is the Jacobian variety of an algebraic curve, its component group has been calculated in many cases. We determine in this paper the component group of the p-new subvariety of $J_0(Mp)$, for $M > 1$ a positive integer and $p \geq 5$ a prime not dividing M. Such a subvariety is not the Jacobian of any obvious curve, but it is not clear if it can ever be realised as the Jacobian of a curve.

1. Introduction

Let A be an abelian variety over $\mathbb{Q}_p$ and let $A_{\mathbb{F}_p}$ denote the special fibre of the Néron model of A over $\mathbb{Z}_p$. Let $A^o_{\mathbb{F}_p}$ denote the connected component of identity in $A_{\mathbb{F}_p}$. The quotient $\Phi(A)_p \stackrel{\text{def}}{=} A_{\mathbb{F}_p}/A^o_{\mathbb{F}_p}$ is a finite group scheme that is étale over $\mathbb{F}_p$. This quotient is the group of connected components (or component group) of $A_{\mathbb{F}_p}$.

When A is the Jacobian variety of an algebraic curve, the component group of A can be, and has been, computed in many cases (see, for example, [2], [10], [3], [9], [7]). However, the methods used in these cases rely heavily on the fact that A is a Jacobian variety. When it is no longer obvious that A is the Jacobian variety of a curve, these known methods cannot be extended and not much is known about the component group of such an abelian variety. In this paper,

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we calculate the component group of some abelian varieties that do not arise naturally as Jacobians of curves and on which the natural polarisation is not principal. However, we are not certain if the abelian varieties considered can be realised as the Jacobians of some curves or if they may admit some principal polarisation.

Let $M > 1$ be a positive integer and let $p \geq 5$ be a prime that does not divide M. Let $J_0(Mp)$ be the Jacobian variety, considered as an abelian variety over $\mathbb{Q}_p$, of the modular curve $X_0(Mp)$. There are two natural degeneracy maps v_1, v_p: $X_0(Mp) \to X_0(M)$. They induce, via Pic functoriality, two maps v_1^*, v_p^*: $J_0(M) \to J_0(Mp)$ on the Jacobian varieties. They also induce, via Albanese functoriality, two maps $(v_1)_*, (v_p)_*$: $J_0(Mp) \to J_0(M)$. The p-new subvariety B of $J_0(Mp)$ is, by definition, the identity component of the intersection of the kernels of $(v_1)_*$ and $(v_p)_*$. Let $\Phi(B)_p$ denote the component group of the special fibre of the Néron model of B over $\mathbb{Z}_p$. We determine $\Phi(B)_p$ and give an application of this knowledge.

THEOREM 1: *If $M > 1$ is a positive integer and $p \geq 5$ is a prime not dividing M, then there is a natural short exact sequence*

$$0 \longrightarrow N \longrightarrow \Phi(B)_p \longrightarrow \Phi_{Mp,p} \longrightarrow 0, \tag{1}$$

compatible with the action of Hecke operators T_n for $n \neq p$, where $\Phi_{Mp,p}$ is the component group of the special fibre of the Néron model of $J_0(Mp)$ over $\mathbb{Z}_p$ and the prime-to-p part of N is

$$N^{(p)} = \ker\left(J_0(M)(\mathbb{F}_{p^2})^{(p)} \xrightarrow{\theta} \mathrm{Hom}_{\overline{\mathbb{F}}_p}(\Sigma(M)(\overline{\mathbb{F}}_p)^{(p)}, \mathbb{G}_m)\right), \tag{2}$$

where θ is a surjective map to be defined in §2, $J_0(M)(\mathbb{F}_{p^2})$ is the group of $\mathbb{F}_{p^2}$-rational points of the abelian variety $J_0(M)$ over $\overline{\mathbb{F}}_p$ and $\Sigma(M)$ is the Shimura subgroup of $J_0(M)$. Moreover, when the p-primary part of $J_0(M)(\mathbb{F}_{p^2})$ is trivial, (2) may be refined to yield

$$N = \ker\left(J_0(M)(\mathbb{F}_{p^2}) \xrightarrow{\theta} \mathrm{Hom}_{\overline{\mathbb{F}}_p}(\Sigma(M)(\overline{\mathbb{F}}_p)^{(p)}, \mathbb{G}_m)\right).$$

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2. Proof of Theorem 1

If A is an abelian variety over $\mathbb{Q}_p$ which has semistable reduction, the connected component of identity $A^o_{\mathbb{F}_p}$ is the extension of an abelian variety by a torus T. In this case, we let

$$X(A) = \mathrm{Hom}_{\overline{\mathbb{F}}_p}(T, \mathbb{G}_m)$$

be its character group. For each A with semistable reduction, there is a monodromy pairing

$$\iota_A\colon X(A) \times X(A') \longrightarrow \mathbb{Z},$$

where A' is the dual abelian variety of A. This pairing may be regarded as an embedding

$$\iota_A\colon X(A') \hookrightarrow X(A)^* \overset{\mathrm{def}}{=} \mathrm{Hom}(X(A), \mathbb{Z}).$$

The cokernel of ι_A is the component group $\Phi(A)_p$. Now let B be the p-new subvariety of $J_0(Mp)$. Then B' is the p-new quotient of $J_0(Mp)$. By Proposition 1 of [6], both B and B' have purely toric reduction. It is well known that $J_0(Mp)$ has semistable reduction at p. Let $f\colon B \to J_0(Mp)$ be the inclusion map. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & X(B') & \overset{\iota_B}{\to} & X(B)^* & \to & \Phi(B)_p & \to 0 \\ (3) \qquad & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & X(J_0(Mp)) & \overset{\iota_J}{\to} & X(J_0(Mp))^* & \to & \Phi_{Mp,p} & \to 0. \end{array}$$

Here ι_B and ι_J are monodromy pairings. The Jacobian $J_0(Mp)$ is identified with its dual. The map $X(B') \to X(J_0(Mp))$ is induced by the map $f'\colon J_0(Mp) \to B'$ dual to f; the map $X(B)^* \to X(J_0(Mp))^*$ is $\mathrm{Hom}(f^*, \mathbb{Z})$, where f^* is the map $f^*\colon X(J_0(Mp)) \to X(B)$; the map $\Phi(B)_p \to \Phi_{Mp,p}$ is f_* induced by f.

According to Proposition 1 of [6], B and $J_0(Mp)$ have the same tori in reduction mod p. Hence the middle vertical map $\mathrm{Hom}(f^*, \mathbb{Z})$ in (3) is an isomorphism. Applying the Snake Lemma to (3), we see that there is an exact sequence of Hecke modules

$$0 \longrightarrow N \longrightarrow \Phi(B)_p \longrightarrow \Phi_{Mp,p} \longrightarrow 0,$$

where

$$N \simeq \mathrm{coker}(X(B') \longrightarrow X(J_0(Mp))). \tag{4}$$

To prove Theorem 1, it suffices to determine N. By definition of B', there is an exact sequence

$$0 \longrightarrow J_{p-\mathrm{old}} \longrightarrow J_0(Mp) \longrightarrow B' \longrightarrow 0, \tag{5}$$

where $J_{p-\mathrm{old}}$ is the p-old subvariety of $J_0(Mp)$. Since $J_0(Mp)$ has semistable reduction with maximal torus $T(Mp)$, by §§7.4 and 7.5 of [2], when we pass to the connected component of identity in the special fibre of the Néron model, the sequence remains exact.

There is also a commutative diagram (cf. [6], proof of Proposition 1)

$$
\begin{array}{cccccccc}
 & 0 \to & H & \to & J_0(M)^2_{\mathbb{F}_p} & \xrightarrow{\alpha} & J_0(M)^2_{\mathbb{F}_p} & \to 0 \\
(6) & & \downarrow & & \downarrow v_1^* \times v_p^* & & \downarrow \mathrm{id} & \\
 & 0 \to & T(Mp) & \to & J_0(Mp)^o_{\mathbb{F}_p} & \to & J_0(M)^2_{\mathbb{F}_p} & \to 0,
\end{array}
$$

where α is the endomorphism of $J_0(M)^2_{\mathbb{F}_p}$ given by $\begin{pmatrix} 1 & V \\ V & 1 \end{pmatrix}$, where V is the Verschiebung endomorphism of $J_0(M)_{\mathbb{F}_p}$, H is the kernel of α and v_1^*, v_p^* are the degeneracy maps. Since α is an isogeny, H is a finite group. Applying the Snake Lemma to (6), it follows that

$$(B')^o \simeq T(Mp)/\tilde{H},$$

where $\tilde{H} = (v_1^* \times v_p^*)(H)$. Applying the functor $X(\cdot) \stackrel{\text{def}}{=} \mathrm{Hom}_{\overline{\mathbb{F}}_p}(\cdot, \mathbb{G}_m)$ to the exact sequence

$$0 \longrightarrow \tilde{H} \longrightarrow T(Mp) \longrightarrow (B')^o \longrightarrow 0,$$

we obtain an exact sequence

$$0 \longrightarrow X(B') \longrightarrow X(J_0(Mp)) \longrightarrow \mathrm{Hom}_{\overline{\mathbb{F}}_p}(\tilde{H}, \mathbb{G}_m) \longrightarrow 0. \tag{7}$$

Note that the sequence (7) is indeed right exact since B' has purely toric reduction and so $\mathrm{Ext}^1((B')^o, \mathbb{G}_m) = 0$. From [12], the kernel of the map $v_1^* \times v_p^*$: $J_0(M)^2 \to J_0(Mp)$ is

$$\left\{ \begin{pmatrix} x \\ -x \end{pmatrix} : x \in \Sigma(M) \right\} \simeq \Sigma(M).$$

The points in the prime-to-p part $\Sigma(M)^{(p)}$ of $\Sigma(M)$ are defined over $\mathbb{Q}_p^{unr}$ and "reduction modulo p" yields an isomorphism $\Sigma(M)(\mathbb{Q}_p^{unr})^{(p)} \simeq \Sigma(M)(\overline{\mathbb{F}}_p)^{(p)}$ ([5], Appendix). Therefore in (6), the prime-to-p part of the kernel of $v_1^* \times v_p^*$ and hence the prime-to-p part of the kernel of $v_1^* \times v_p^* : H \to T(Mp)$ are isomorphic to $\Sigma(M)(\overline{\mathbb{F}}_p)^{(p)}$.

It is routine to check that H may be identified with $J_0(M)[1-V^2]$ via the map

$$
\begin{array}{ccc}
J_0(M)[1-V^2] & \longrightarrow & J_0(M)^2\left[\begin{pmatrix} 1 & V \\ V & 1 \end{pmatrix}\right] \\
x & \mapsto & \begin{pmatrix} x \\ -Vx \end{pmatrix}.
\end{array}
$$

(For an abelian variety A and an endomorphism ϕ of A, $A[\phi]$ denotes the kernel of ϕ in A.) Taking $H = J_0(M)[1-V^2]$, it follows that its Cartier dual $X(H) = J_0(M)[1-F^2] \simeq J_0(M)(\mathbb{F}_{p^2})$.

Applying the $X(\cdot)$ functor to the exact sequence

$$0 \longrightarrow \Sigma(M)(\overline{\mathbb{F}}_p)^{(p)} \longrightarrow H^{(p)} \longrightarrow \tilde{H}^{(p)} \longrightarrow 0, \tag{8}$$

we obtain

$$0 \longrightarrow X(\tilde{H}^{(p)}) \longrightarrow J_0(M)(\mathbb{F}_{p^2})^{(p)} \longrightarrow X(\Sigma(M)(\overline{\mathbb{F}}_p)^{(p)}) \longrightarrow 0. \tag{9}$$

This sequence is again right exact, this time by cardinality consideration. Combining (4), (7) and (9), we have

$$N^{(p)} = \ker\left(J_0(M)(\mathbb{F}_{p^2})^{(p)} \to \mathrm{Hom}_{\overline{\mathbb{F}}_p}(\Sigma(M)(\overline{\mathbb{F}}_p)^{(p)}, \mathbb{G}_m)\right).$$

When H has trivial p-primary part, $H^{(p)}$ and $\tilde{H}^{(p)}$ in (8) can be replaced with H and $\tilde{H}$. Retracing the proof above shows that

$$N = \ker\left(J_0(M)(\mathbb{F}_{p^2}) \to \mathrm{Hom}_{\overline{\mathbb{F}}_p}(\Sigma(M)(\overline{\mathbb{F}}_p)^{(p)}, \mathbb{G}_m)\right).$$

The compatibility of (8) with the action of the Hecke operators is clear from the proof above.

3. Examples

The structure of the component group $\Phi_{Mp,p}$ is given in [3], while that of the Shimura subgroup $\Sigma(M)$ is given in [8]. If $J_0(M)(\mathbb{F}_{p^2})$ is known, then at least the prime-to-p part of $\Phi(B)_p$ can be fully determined. We give two examples here.

Example 1: When $p = 31$ and $M = 11$.

From [3], it follows easily that

$$\Phi_{11\cdot 31,31} \simeq \mathbb{Z}/10\mathbb{Z}.$$

Since the genus of $X_0(11)$ is one, so is the $\mathbb{Z}$-rank of its Hecke algebra. From the tables in [1], we have that $T_{31} = 7$. Therefore, $J_0(11)(\mathbb{F}_{31^2})$ is a group of order $(1+31)^2 - 7^2 = 3\cdot 5^2 \cdot 13$. In particular, its order is coprime to 31. From [8], it follows easily that the order of $\Sigma(11)$ is 5. Combining all the above information, we see that $\Phi(B)_p$ (when $p = 31$ and $M = 11$) is a group of order $2\cdot 3\cdot 5^2\cdot 13$.

Example 2: When $p = 5$ and $M = 37$.

From [3], it follows that

$$\Phi_{37\cdot 5,5} \simeq \mathbb{Z}/2\cdot 3\cdot 19\mathbb{Z}.$$

The genus of $X_0(37)$ is 2, hence the $\mathbb{Z}$-rank of the Hecke algebra is also 2. The abelian surface $J_0(37)$ is $\mathbb{Q}$-isogenous to the product of two elliptic curves with traces of Frobenius over $\mathbb{F}_5$ equal to 0 and 2. Hence the traces of Frobenius over $\mathbb{F}_{5^2}$ are $0^2 - 2\cdot 5 = -10$ and $2^2 - 2\cdot 5 = -6$. Correspondingly, the orders of the two elliptic curves over $\mathbb{F}_{25}$ are $5^2 + 1 - (-10) = 36$ and $5^2 + 1 - (-6) = 32$. Hence the number of points on any abelian surface over $\mathbb{F}_{25}$ isogenous over this field to this product of elliptic curves is $32\cdot 36 = 2^7\cdot 3^2$. By [8], $\Sigma(37)$ is of order 3. Therefore, $\Phi(B)_p$ (when $p = 5$ and $M = 37$) is a group of order $2^8\cdot 3^2\cdot 19$.

4. An application

We continue to assume that $M > 1$ is a positive integer and $p \geq 5$ is a prime not dividing M. Let γ be the map

$$\gamma = v_1^* \times v_p^* \times \cdots \times v_{p^{r-1}}^* \colon J_0(Mp)^r \longrightarrow J_0(Mp^r), \quad r \geq 2.$$

In [6], it was proved that there is a natural inclusion of the group $(\ker\gamma) \cap B^r$ into the kernel of $\tilde{\gamma} : \Phi(B)_p^r \to \Phi_{Mp^r,p}$, where $\tilde{\gamma}$ is the map induced from the restriction of γ to B^r and $\Phi_{Mp^r,p}$ is the component group of the special fibre of the Néron model of $J_0(Mp^r)$ over $\mathbb{Z}_p$. Theorem 1 enables us to determine the prime-to-p part of $\ker\tilde{\gamma}$.

THEOREM 2: *Let $M > 1$ be an integer and let $p \geq 5$ be a prime not dividing M. Let B denote the p-new subvariety of $J_0(Mp)$ and let γ be the map*

$$\gamma = v_1^* \times v_p^* \times \cdots \times v_{p^{r-1}}^* \colon J_0(Mp)^r \longrightarrow J_0(Mp^r), \quad r \geq 2.$$

Let $\ker\tilde{\gamma}$ be the kernel of $\tilde{\gamma}$: $\Phi(B)_p^r \to \Phi_{Mp^r,p}$, induced by the restriction of γ to B^r. Then there is a short exact sequence

$$0 \longrightarrow N^r \longrightarrow \ker\tilde{\gamma} \longrightarrow K_\Phi \longrightarrow 0,$$

where N is as in Theorem 1 and K_Φ is the kernel of the induced map

$$v_1^* \times v_p^* \times \cdots \times v_{p^{r-1}}^* \colon \Phi_{Mp,p}^r \longrightarrow \Phi_{Mp^r,p}.$$

Proof: It suffices to consider the commutative diagram

$$
(10)\qquad
\begin{array}{ccccccccc}
0 \longrightarrow & N^r & \longrightarrow & \Phi(B)_p^r & \longrightarrow & \Phi_{Mp,p}^r & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & 0 & \longrightarrow & \Phi_{Mp^r,p} & \xrightarrow{\mathrm{id}} & \Phi_{Mp^r,p} & \longrightarrow 0,
\end{array}
$$

where the top exact sequence is just r copies of (1), while the vertical maps are $v_1^* \times \cdots \times v_{p^{r-1}}^*$. Applying once again the Snake Lemma to (10), we get

$$0 \longrightarrow N^r \longrightarrow \ker\tilde{\gamma} \longrightarrow K_\Phi \longrightarrow 0. \qquad \blacksquare$$

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